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Invariant measure on sums of symmetric 3 × 3 matrices with specified eigenvalues

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Abstract. Suppose $\mathbf{A}$ and $\mathbf{B}$ are symmetric 3×3 matrices. Rotate one of them by $\sigma \in \mathrm{SO}(3)$ and add; the eigenvalues of the resulting matrix $\mathbf{A} + \mathbf{B}^{\sigma}$ vary over a set determined by the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. In the case where one of $\mathbf{A}$ and $\mathbf{B}$ has a repeated eigenvalue we find the image of SO(3) Haar measure $\mathrm{d}\sigma$ on this set, which describes the coupling of different rigid rotors.

1. Introduction

Several authors have considered the question of describing the possible eigenvalues of $\mathbf{A} + \mathbf{B}$, if $\mathbf{A}$ and $\mathbf{B}$ are symmetric $n \times n$ matrices with specified eigenvalues (see Horn 1962, Lidskii 1982, Thompson 1986). An equivalent formulation is to let $\mathbf{A}$ and $\mathbf{B}$ be diagonal and consider $\mathbf{A}+\mathbf{B}^{\sigma}$, where the superscript means conjugation by $\sigma \in \mathrm{SO}(n)$.

In this paper we deal with the case of $n = 3$ and consider the related question of finding the image of the SO(3) Haar measure on the space of sets of eigenvalues of the sum matrix, subject to the condition that one of the matrices $\mathbf{A}$ and $\mathbf{B}$ has a repeated eigenvalue.

This condition simplifies the problem because the parameter σ can be taken to be in $\mathrm{SO}(2)\backslash\mathrm{SO}(3)$. Observing that the question is essentially unchanged by adding a scalar matrix to shift the eigenvalues, we can assume that the repeated eigenvalue is 0 and that the trace of $\mathbf{A} + \mathbf{B}$ is zero. Multiplying by a scalar we can assume that $\mathbf{B} = \mathrm{diag}(1, 0, 0)$.

A generic (tri-axial) irreducible representation of the rotor group $[\mathbb{R}^5]\mathrm{SO}(3)$ is induced from a one-dimensional representation of the subgroup $[\mathbb{R}^5]D_2$. The characters of $\mathbb{R}^5$ can be interpreted as sets of quadruple moments, given as symmetric 3×3 matrices, and the induced representation is independent of conjugation by SO(3) (see Ui 1970, Weaver *et al* 1973, Rowe *et al* 1989).

If two rigid rotors are coupled then the resulting quadrupole moments are found by adding the corresponding symmetric matrices, and they depend on the relative orientation of the original bodies. The measure we find in equation (4.2) gives the probabilistic weight with which each resulting rotor occurs, assuming random alignment of the original rotors. In this way it is related to the problem of decomposing the tensor product of representations of the rigid rotor algebra $[\mathbb{R}]^5\mathrm{SO}(3)$.

2. Eigenvalues of the sum of two matrices

We begin with the following easy lemma; a proof is included for completeness.

Lemma 2.1. Suppose $\mathbf{A}$ is an $n \times n$ Hermitian matrix and $\mathbf{P}$ a positive $n \times n$ matrix (i.e. $\langle Pv, v\rangle \geq 0$, for every $v \in \mathbb{C}^n$). If the eigenvalues of $\mathbf{A}$ are $a_1 \geq a_2 \geq \cdots \geq a_n$ and the eigenvalues of $\mathbf{A}+\mathbf{P}$ are $s_1 \geq s_2 \geq \cdots \geq s_n$, then

$$s_k \geq a_k$$

for each k. (Adding a positive matrix causes the eigenvalues of a Hermitian matrix to 'increase'.)

Proof. Suppose $s_k < a_k$, for some k. Consider the matrix $\mathbf{A} - a_k\mathbf{I}$. The sum of the $\mathbf{A}$-eigenspaces corresponding to the eigenvalues $a_1, \ldots, a_k$ is an $\mathbf{A}$-invariant subspace V on which $\mathbf{A} - a_k\mathbf{I} \geq 0$. The dimension of $\mathbf{V}$ is at least k.

On the other hand, the sum of the $(\mathbf{A}+\mathbf{P})$ eigenspaces corresponding to the eigenvalues $s_k, s_{k+1}, \ldots, s_n$ is an $(\mathbf{A}+\mathbf{P})$-invariant subspace W on which $\mathbf{A}+\mathbf{P}-a_k\mathbf{I}$ is negative definite. The dimension of W is at least $n-k+1$.

Because of these dimensions, it is possible to find a non-zero $v \in V \cap W$. So

$$0 > \langle(\mathbf{A}+\mathbf{P}-a_k\mathbf{I})v, v\rangle \geq \langle(\mathbf{A}-a_k\mathbf{I})v, v\rangle \geq 0$$

and the lemma is proved. □

Theorem 2.2. Suppose $\mathbf{A}$ and $\mathbf{B}$ are symmetric real 3×3 matrices, and that the eigenvalues of $\mathbf{A}$ are $a \geq b \geq c$ and the eigenvalues of $\mathbf{B}$ are $r \geq s \geq t$. If the eigenvalues of $\mathbf{A}+\mathbf{B}$ are $\alpha \geq \beta \geq \gamma$, then

$$\max(a+t, b+s, c+r) \leq \alpha \leq a+r$$

$$\max(b+t, c+s,) \leq \beta \leq \min(a+s, b+r)$$

$$c+t \leq \gamma \leq \min(a+t, b+s, c+r).$$

Proof. First,

$$\alpha = \sup_{\|v\|=1} \langle(\mathbf{A}+\mathbf{B})v, v\rangle \leq \sup_{\|v\|=1} \langle\mathbf{A}v, v\rangle + \sup_{\|v\|=1} \langle\mathbf{B}v, v\rangle = a+r.$$

Secondly,

$$\alpha = \sup_{\|v\|=1} \langle(\mathbf{A}+\mathbf{B})v, v\rangle \geq \sup_{\|v\|=1} \langle\mathbf{A}v, v\rangle + \inf_{\|v\|=1} \langle\mathbf{B}v, v\rangle = a+t.$$

Similarly $\alpha \geq c+r$.

If we let $\mathbf{P}$ be the projection onto the a-eigenspace of $\mathbf{A}$, then $\mathbf{P} \geq 0$. The matrix $\mathbf{A}-(a-b)\mathbf{P}$ has its b-eigenspace of dimension at least 2. Similarly, if $\mathbf{Q}$ is the projection onto the r-eigenspace of $\mathbf{B}$, then $\mathbf{B}-(r-s)\mathbf{Q}$ has an s-eigenspace of dimension at least 2.

The intersection of these two eigenspaces must be non-trivial, which means that $b+s$ is an eigenvalue of $[\mathbf{A}-(a-b)\mathbf{P}]+[\mathbf{B}-(r-s)\mathbf{Q}]$. Since $\mathbf{A}+\mathbf{B} \geq [\mathbf{A}-(a-b)\mathbf{P}]+[\mathbf{B}-(r-s)\mathbf{Q}]$ the largest eigenvalue α of $\mathbf{A}+\mathbf{B}$ must satisfy $\alpha \geq b+s$.

This completes the inequalities for α, and the inequalities for γ are equivalent (by considering $-\mathbf{A}-\mathbf{B}$, for instance).

Using lemma 2.1 we observe that since $\mathbf{A} \leq a\mathbf{I}$ the middle eigenvalue β of $\mathbf{A}+\mathbf{B}$ must be less than or equal to the middle eigenvalue of $a\mathbf{I}+\mathbf{B}$, which is $a+s$. Similarly $\beta \leq b+r$, so $\beta \leq \min(a+s, b+r)$. The lower bound for β is obtained in the same way (or by considering $-\mathbf{A}-\mathbf{B}$). □

3. Barycentric coordinates and rotation parameters

In fact, if (α, β, γ) satisfy the inequalities of theorem 2.2 then they are the eigenvalues of a matrix of the form $\mathbf{A}+\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ have the specified eigenvalues (cf Lidskii 1982). However we will only prove this in a special case.

It is convenient to plot the eigenvalues (α, β, γ) of a matrix $\mathbf{S}$ using barycentric coordinates. Assuming $\alpha \geq \beta \geq \gamma$, we associate to (α, β, γ) the point

$$(x, y) = \left(\alpha - \frac{1}{2}(\beta+\gamma), \frac{\sqrt{3}}{2}(\beta-\gamma)\right)$$

in the plane. This representation has the advantage that it is unaffected by the addition of a scalar matrix.

In particular, we can assume $\alpha+\beta+\gamma=0$. In this case $x = \frac{3}{2}\alpha$ and along the other barycentric axes we find that

$$\tfrac{1}{2}(x-\sqrt{3}y) = -\tfrac{3}{2}\beta \qquad \text{and} \qquad \tfrac{1}{2}(x+\sqrt{3}y) = -\tfrac{3}{2}\gamma\ . \tag{3.1}$$

Since this allows us to write the eigenvalues α, β, γ of $\mathbf{S}$ in terms of (x, y), we can easily compute the invariants of $\mathbf{S}$.

Lemma 3.1. If $\alpha \geq \beta \geq \gamma$ satisfying $\alpha+\beta+\gamma=0$ are the eigenvalues of $\mathbf{S}$, then

$$\begin{aligned} \det(\mathbf{S}) &= \frac{2x}{3}\left(\frac{x^2}{9} - \frac{y^2}{3}\right) \\ \sigma_2(\mathbf{S}) &= \alpha\beta+\alpha\gamma+\beta\gamma = -\tfrac{1}{3}(x^2+y^2). \end{aligned} \tag{3.2}$$

Now we want to consider the matrix

$$\mathbf{S}(\sigma) = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}^{\sigma} \tag{3.3}$$

where the superscript indicates conjugation by $\sigma \in \mathrm{SO}(3)$ and for convenience $a+b+c+1=0$. Writing

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{3.4}$$

we see that $\mathbf{S}(\sigma)$ is independent of ψ, and indeed

$$S(\sigma) = \begin{pmatrix} \cos^2\theta + a & -\cos\theta\sin\theta\cos\phi & \cos\theta\sin\theta\sin\phi \\ -\cos\theta\sin\theta\cos\phi & \sin^2\theta\cos^2\phi + b & -\sin^2\theta\cos\phi\sin\phi \\ \cos\theta\sin\theta\sin\phi & -\sin^2\theta\cos\phi\sin\phi & \sin^2\theta\sin^2\phi + c \end{pmatrix}. \tag{3.5}$$

It is straightforward to calculate the determinant and the two-elementary symmetric function σ_2 of this matrix, and we find

$$\begin{aligned} \det(\mathbf{S}(\sigma)) &= abc + bc\cos^2\theta + ab\sin^2\theta\sin^2\phi + ac\sin^2\theta\cos^2\phi \\ \sigma_2(\mathbf{S}(\sigma)) &= a\sin^2\theta + b(\cos^2\theta + \sin^2\theta\sin^2\phi) + c(\cos^2\theta + \sin^2\theta\cos^2\phi) \\ &\quad + ab + bc + ac \end{aligned} \tag{3.6}$$

Now we ask for what $\sigma \in \mathrm{SO}(3)$ will the matrix $\mathbf{S}(\sigma)$ given by (3.5) have eigenvalues α, β, γ. Since the trace of $\mathbf{S}(\sigma)$ is zero, this amounts to asking when $\det(\mathbf{S}(\sigma))$ and $\sigma_2(\mathbf{S}(\sigma))$ equal the corresponding invariants of the matrix $\mathrm{diag}(\alpha, \beta, \gamma)$, and these latter have been expressed in terms of (x, y) in lemma 3.1.

The conditions are

$$\begin{aligned} &abc + bc\cos^2\theta + ab\sin^2\theta\sin^2\phi + ac\sin^2\theta\cos^2\phi = \frac{2x}{3}\left(\frac{x^2}{9} - \frac{y^2}{3}\right) \\ &a\sin^2\theta + b(\cos^2\theta + \sin^2\theta\sin^2\phi) + c(\cos^2\theta + \sin^2\theta\cos^2\phi) \\ &\qquad + ab + bc + ac = -\tfrac{1}{3}(x^2 + y^2). \end{aligned} \tag{3.7}$$

We solve and find that

$$\begin{aligned} \cos^2\theta &= \frac{1}{(a-b)(a-c)}\left[\frac{2}{27}x^3 - \frac{2}{9}xy^2 + \frac{a}{3}\left(x^2 + y^2\right) - a^3\right] \\ &= \frac{1}{27(a-b)(a-c)}(2x - 3a)(x - \sqrt{3}y + 3a)(x + \sqrt{3}y + 3a) \end{aligned} \tag{3.8}$$

$$\begin{aligned} \sin^2\phi &= \frac{a-b}{b-c}\,\frac{-2x^3 + 6xy^2 - 9c(x^2+y^2) + 27c^3}{2x^3 - 6xy^2 + 9a(x^2+y^2) + 27(a+a^2)(b+c) - 27bc} \\ &= \frac{a-b}{b-c}\,\frac{-(2x-3c)(x+\sqrt{3}y+3c)(x-\sqrt{3}y+3c)}{(2x-3a)(x+\sqrt{3}y+3a)(x-\sqrt{3}y+3a) - 27(a-b)(a-c)}\,. \end{aligned} \tag{3.9}$$

Using (3.1) we express the inequalities of theorem 2.2 in terms of x and y. We use $r = 1$, $s = t = 0$ and assume $a + b + c + 1 = 0$. The inequalities are

$$\begin{array}{lclcl} \frac{3}{2}\max(a, c+1) & \leq & x & \leq & \frac{3}{2}(a+1) \\ \frac{3}{2}b & \leq & \frac{1}{2}(\sqrt{3}y - x) & \leq & \frac{3}{2}\min(a, b+1) \\ \frac{3}{2}c & \leq & -\frac{1}{2}(x + \sqrt{3}y) & \leq & \frac{3}{2}\min(b, c+1). \end{array} \tag{3.10}$$

Now suppose x and y satisfy these inequalities. To show that it is possible to find $\sigma \in \mathrm{SO}(3)$ so that the eigenvalues of $\mathbf{S}(\sigma)$ correspond to the point (x, y), we must first show that it is possible to solve for θ using (3.8). To do this it is necessary and sufficient to know that the right-hand side of (3.8) lies between 0 and 1.

First we note that from (3.10), $x \geq \frac{3}{2}a$, so $2x - 3a \geq 0$. Secondly, $\frac{1}{2}(\sqrt{3}y - x) \leq \frac{3}{2}a$, so $x - \sqrt{3}y \geq -3a$ and $x - \sqrt{3}y + 3a \geq 0$. Similarly $-\frac{1}{2}(x + \sqrt{3}y) \leq \frac{3}{2}b$, so $x + \sqrt{3}y \geq -3b$ and $x + \sqrt{3}y + 3a \geq 3a - 3b \geq 0$. From this we conclude that the right side of (3.8) is non-negative.

On the other hand, from (3.10) we see that $x \leq \frac{3}{2}(a+1)$, so $2x - 3a \leq 3$. Also, since $\frac{1}{2}(x - \sqrt{3}y) \leq -\frac{3}{2}b$, we have $x - \sqrt{3}y + 3a \leq 3(a - b)$, and since $\frac{1}{2}(x + \sqrt{3}y) \leq -\frac{3}{2}c$, we have $x + \sqrt{3}y + 3a \leq 3(a - c)$. Together these inequalities show that the right side of (3.8) cannot exceed 1.

We conclude that (3.8) determines a unique value of $\theta \in [0, \frac{\pi}{2}]$ for every (x, y) satisfying (3.10).

It is slightly more difficult to show that the expression (3.9) for $\sin^2\phi$ is between 0 and 1.

Theorem 3.2. If (x, y) lies in the region specified by (3.10), then the right-hand side of (3.9) is between 0 and 1, so it is possible to solve for $\phi \in [0, \frac{\pi}{2}]$.

Proof. It is easy to check that for $\frac{3}{2}a < x < \frac{3}{2}(a+1)$, the right-hand side of (3.9) equals 0 when $x + \sqrt{3}y + 3c = 0$ and equals 1 when $x - \sqrt{3}y + 3b = 0$. These are the upper and lower edges respectively of the region specified by (3.10).

Fixing x with $\frac{3}{2}a < x < \frac{3}{2}(a+1)$ and regarding the right-hand side of (3.9) as a function of y, we see it is of the form

$$C\,\frac{y^2 - K^2}{L - y^2} \tag{3.11}$$

where C, K, L are constants and $C > 0$.

To show that it lies between 0 and 1 on the specified region we will show it decreases from 1 on $x - \sqrt{3}y + 3b = 0$ to 0 on $x + \sqrt{3}y + 3c = 0$.

The function (3.11) may have singularities (at $y = \pm\sqrt{L}$ if $L \geq 0$), but on any interval in $[0, \infty)$ which does not contain a singularity, it is monotonic (since the sign of the derivative does not change).

So it will suffice to show that the denominator is non-zero on the specified region, since (3.11) must then be decreasing from 1 on the bottom edge to 0 on the top edge.

On the top edge $x + \sqrt{3}y + 3c = 0$ with $\frac{3}{2}a < x < \frac{3}{2}(a+1)$, the denominator of (3.9) is a positive constant times

$$(\tfrac{2}{3}x - a)(a - c)(\tfrac{2}{3}x + a + c) - 2a^2 - a - bc.$$

This is a parabola opening upwards. At the right endpoint $x = \frac{3}{2}(a+1)$ we use the relation $a + b + c + 1 = 0$ to find the denominator equals 0. At the left endpoint $x = \frac{3}{2}a$ the denominator equals $-2a^2 - a - bc = -(a - b)(a - c)$, which is negative. We conclude that the denominator is negative for all x with $\frac{3}{2}a < x < \frac{3}{2}(a+1)$.

On the bottom edge $x - \sqrt{3}y + 3b = 0$ the denominator of (3.9) is a positive constant times

$$(\tfrac{2}{3}x - a)(\tfrac{2}{3}x + a + b)(a - b) - 2a^2 - a - bc.$$

Once again we find this is negative at $x = \frac{3}{2}a$ and zero at $x = \frac{3}{2}(a+1)$.

The denominator of (3.11), as a function of y with x fixed, is a downward-opening parabola with vertex at $y = 0$. It is negative on both $x + \sqrt{3}y + 3c = 0$ and $x - \sqrt{3}y + 3b = 0$ for $\frac{3}{2}a < x < \frac{3}{2}(a+1)$, so it must be negative between them and therefore negative on the region (3.10), which lies inside the triangle bounded by $x + \sqrt{3}y + 3c = 0$, $x - \sqrt{3}y + 3b = 0$ and $x = \frac{3}{2}a$. In particular it is not zero there. □

For each (x, y) in the polygonal region determined by (3.10) the formulae (3.8) and (3.9) define $\theta \in [0, \frac{\pi}{2}]$ and $\phi \in [0, \frac{\pi}{2}]$. Except on some of the boundaries θ and ϕ are uniquely determined. Ignoring this null set, it is possible to express θ and ϕ as functions of (x, y).

4. The invariant measure

For our purposes the normalized Haar measure $d\sigma$ on $\mathrm{SO}(2)\backslash\mathrm{SO}(3)$ can be expressed in terms of the parametrization (3.4) as

$$d\sigma = \frac{2}{\pi}\sin\theta\, d\theta\, d\phi$$

with $\theta, \phi \in [0, \frac{\pi}{2}]$.

We want to consider the image of $\mathrm{d}\sigma$ on the space of eigenvalues and express it in terms of the coordinates (x, y).

Using the intermediate variables $(\sigma_2, \det)$, the two-elementary symmetric function and determinant of the matrix $\mathbf{S}(\sigma)$, we can write

$$\mathrm{d}\sigma = \frac{2}{\pi}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{2}{\pi}\sin\theta \cdot \left|\frac{\partial(\sigma_2, \det)}{\partial\theta\;\partial\phi}\right|^{-1} \cdot \left|\frac{\partial(\sigma_2, \det)}{\partial x\;\partial y}\right| \mathrm{d}x\,\mathrm{d}y. \tag{4.1}$$

Using (3.6) it is easy to evaluate the first Jacobian determinant; we find

$$\left|\frac{\partial(\sigma_2, \det)}{\partial\theta\;\partial\phi}\right| = 4(a-b)(a-c)(b-c)\cos\theta\,\sin^3\theta\,\cos\phi\,\sin\phi.$$

From lemma 3.1 the other determinant is

$$\left|\frac{\partial(\sigma_2, \det)}{\partial x\;\partial y}\right| = \frac{4}{27}(3x^2y - y^3).$$

Substituting in (4.1), we obtain

$$\mathrm{d}\sigma = \frac{1}{2\pi(a-b)(a-c)(b-c)} \cdot \frac{1}{\cos\theta\,\sin^2\theta\,\cos\phi\,\sin\phi} \cdot \frac{4}{27}(3x^2y - y^3)\mathrm{d}x\,\mathrm{d}y.$$

Substituting the expressions from (3.8) and (3.9) and simplifying, we obtain

$$\begin{aligned} d\sigma = \frac{6\sqrt{3}}{\pi} &\left|(2x-3a)(x+\sqrt{3}y+3a)(x-\sqrt{3}y+3a)\right|^{-1/2} \\ \times &\left|(2x-3b)(x+\sqrt{3}y+3b)(x-\sqrt{3}y+3b)\right|^{-1/2} \\ \times &\left|(2x-3c)(x+\sqrt{3}y+3c)(x-\sqrt{3}y+3c)\right|^{-1/2}\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.2}$$

5. Geometrical considerations

It is interesting to interpret (4.2) in terms of the action of the GL(3) Weyl group on (x, y) (i.e. the permutation action of $W = S_3$ on triples of eigenvalues).

The line $x - \sqrt{3}y + 3a = 0$ is the reflection of $x + \sqrt{3}y + 3a = 0$ in the Weyl wall $y = 0$ and is also the reflection of $2x - 3a = 0$ in the Weyl wall $\sqrt{3}x - y = 0$.

This means that the factor $|(2x-2a)(x+\sqrt{3}y+3a)(x-\sqrt{3}y+3a)|$ is W-invariant, and similarly for the other factors in (4.2). The weight function blows up as (x, y) approaches certain sides of the polygon determined by (3.10), with a $(-\frac{1}{2})$-power singularity. Some of the factors in the weight function provide this singularity and others are there 'for symmetry'.

6. Two axially symmetric rotors

In the preceding discussion we have assumed that one matrix has distinct eigenvalues a, b, c and the other has one double eigenvalue. If both matrices have a double eigenvalue, the case which arises from the moments of inertia of two axially symmetric objects, then the question is simpler.

Suppose the repeated eigenvalue is 0 for each matrix (by adding scalars as needed). If $\mathbf{A} = \mathrm{diag}\,(a, 0, 0)$ and $\mathbf{B} = \mathrm{diag}\,(b, 0, 0)$, let

$$\mathbf{S}(\sigma) = \mathbf{A} + \mathbf{B}^{\sigma}$$

with $\sigma \in \mathrm{SO}(3)$ given by (3.4). Then the invariants of $\mathbf{S}(\sigma)$ are independent of ϕ and ψ.

Since $\mathbf{A}$ and $\mathbf{B}$ are both of rank 1 we see that 0 is always an eigenvalue of $\mathbf{S}(\sigma)$. The other two eigenvalues add up to $a + b$.

Multiplying by -1 if necessary, we assume $a > 0$ and $a \geq |b|$. There are two cases to distinguish: case 1, in which $b > 0$, and case 2, in which $b < 0$. In case 1 the largest possible eigenvalue is obtained when the axes of symmetry are aligned and in case 2 it is obtained when they are orthogonal.

In case 1 it is easy to check that as σ ranges over SO(3) the largest eigenvalue λ of $\mathbf{S}(\sigma)$ ranges over $[a, a+b]$ (and the middle eigenvalue $a + b - \lambda$ ranges over $[0, b]$).

The normalized Haar measure on $\mathrm{SO}(2)\backslash\mathrm{SO}(3)/\mathrm{SO}(2)$ is $\mathrm{d}\sigma = \sin\theta \mathrm{d}\theta$, for $\theta \in [0, \frac{\pi}{2}]$ in the parametrization (3.4). It is easy to show that its image in terms of the largest eigenvalue λ is

$$\mathrm{d}\sigma = \frac{\lambda - [(a+b)/2]}{\sqrt{ab(\lambda - a)(\lambda - b)}}\, \mathrm{d}\lambda. \tag{6.1}$$

In case 2, with $a > 0 > b$, $a \geq |b|$, the largest eigenvalue λ of $\mathbf{S}(\sigma)$ ranges over $[a + b, a]$, the middle eigenvalue is 0 and the smallest eigenvalue $a + b - \lambda$ ranges over $[b, 0]$. The Haar measure in terms of the largest eigenvalue λ is

$$\mathrm{d}\sigma = \frac{\lambda - [(a+b)/2]}{\sqrt{|ab|(a - \lambda)(\lambda - b)}}\, \mathrm{d}\lambda. \tag{6.2}$$

Once again there is a singularity $|\lambda - a|^{-1/2}$ at one boundary of the interval in each of (6.1) and (6.2). The other 'singularity' $|\lambda - b|^{1/2}$ is its Weyl reflection (the Weyl group reflects in $(a + b)/2$, so the absolute value of the numerator is also invariant).

It is interesting to interpret the difference between (6.1) and (6.2) geometrically. In each case the weight function blows up at one end of the interval and not the other. Roughly speaking, aligning the two axes of symmetry is rare (one degree of freedom in SO(3)) compared with having the axes orthogonal (two degrees of freedom in SO(3)). The measure weights move heavily the eigenvalue $\lambda = a$ corresponding to orthogonal axes than the opposite end of the interval which corresponds to aligned axes.

Of course (6.1) and (6.2) can both be written as

$$\mathrm{d}\sigma = \frac{\lambda - [(a+b)/2]}{\sqrt{|ab| \cdot |\lambda - a| \cdot |\lambda - b|}}\, \mathrm{d}\lambda. \tag{6.3}$$

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